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PARTIAL DIFFERENTIAL EQUATIONS: - BASIC CONCEPTS EXAMPLES AND ONE METHOD OF SEPARATION OF VARIABLES

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ABSTRACT

This technique may be used to a broad variety of issues in mathematical physics, applied mathematics, and engineering science. These are the kinds of questions that fall into the "class of problems" category. In the next section, we will first introduce the technique of separation of variables, and then we will investigate the circumstances under which this approach may be used to solve problems involving two independent variables and second-order partial differential equations. The second-order homogeneous partial differential equation is the focus of our attention here.

Keywords: *Separation, Variables, Differential Equations*

INTRODUCTION

When solving initial boundary-value problems involving linear partial differential equations, a common approach is to combine the technique of separating variables with the idea of superposition. This approach has found widespread use. The dependent variable, denoted by u (x, y), is often written down using the separable form. $u(x, y) = X(x)Y(y)$, are functions, respectively, of the variables x and y. There are several situations in which the partial differential equation may be simplified into two ordinary differential equations for X and Y. Equations with three or more independent variables may be solved using a method that is similar to this one. Nevertheless, the issue of whether or not a partial differential equation can be broken down into two or more ordinary differential equations is not at all an easy one to answer. Despite the existence of this concern, the approach is commonly used in the process of finding solutions to a significant number of initial boundary-value problems. Another name for this approach to problem solving is the Fourier technique, and it's also sometimes called the method of eigenfunction expansion. Therefore, the process that was stated above leads to the significant concepts of eigenvalues, eigenfunctions, and orthogonality, all of which are quite generic and strong when it comes to dealing with linear issues. The following examples demonstrate the comprehensive character of this approach to finding a solution.

Separation of Variables

For the purpose of resolving initial boundary-value issues, we will discuss the technique of separation of variables, which is among the most typical and fundamental approaches to the problem-solving process. This technique may be used to a broad variety of issues in mathematical physics, applied mathematics, and engineering science. These are the kinds of questions that fall into the "class of problems" category. In the next section, we will first introduce the technique of separation of variables, and then we will investigate the circumstances under which this approach may be used to solve problems involving two independent variables and second-order partial differential equations. The second-order homogeneous partial differential equation is the focus of our attention here.

$$a^* u_{x^* x^*} + b^* u_{x^* y^*} + c^* u_{y^* y^*} + d^* u_{x^*} + e^* u_{y^*} + f^* u = 0$$

$$x = x(x^*, y^*), \qquad y = y(x^*, y^*),$$

Where

$$\frac{\partial(x, y)}{\partial(x^*, y^*)} \neq 0,$$

We always have the option of transforming the equation into its canonical form.

$$a(x, y) u_{xx} + c(x, y) u_{yy} + d(x, y) u_x + e(x, y) u_y + f(x, y) u = 0, (7.2.3)$$

where X and Y are functions of x alone and functions of y alone, respectively, and where both sets of functions are continuously differentiable. By plugging equations into, we are able to get.

$$a X''Y + c XY'' + d X'Y + e XY' + f XY = 0,$$

where the primes signify difference with regard to the factors that are relevant. Let us assume that there is a function p (x, y) such that, if we divide the equation by p, we get the following.

$$a_1(x) X''Y + b_1(y) XY'' + a_2(x) X'Y + b_2(y) XY' + [a_3(x) + b_3(y)] XY = 0.$$

When we divide the equation one again by XY, we get the following.

$$\left[a_1 \frac{X''}{X} + a_2 \frac{X'}{X} + a_3\right] = -\left[b_1 \frac{Y''}{Y} + b_2 \frac{Y'}{Y} + b_3\right].$$

where A, B, C, D, E, and F are all non-zero constants that are being used. As we did previously, we are going to assume that there is a separateable answer in the form of.

$$u(x, y) = X(x) Y(y) \neq 0.$$

When we plug this into the equation, we get the following.

$$AX''Y + BX'Y' + CXY'' + DX'Y + EXY' + FXY = 0.$$

The solution to this problem may be found by dividing it by AXY.

$$\frac{X''}{X} + \frac{B}{A}\frac{X'}{X}\frac{Y'}{Y} + \frac{C}{A}\frac{Y''}{Y} + \frac{D}{A}\frac{X'}{X} + \frac{E}{A}\frac{Y'}{Y} + \frac{F}{A} = 0, \quad A \neq 0.$$

In order to determine the solution, we differentiate this equation with respect to x.

$$\left(\frac{X''}{X}\right)' + \frac{B}{A}\left(\frac{X'}{X}\right)'\frac{Y'}{Y} + \frac{D}{A}\left(\frac{X'}{X}\right)' = 0.$$

Thus, we have

$$\frac{\left(\frac{X''}{X}\right)'}{\frac{B}{A}\left(\frac{X'}{X}\right)'} + \frac{D}{B} = -\frac{Y'}{Y}.$$

Because it is evident that this equation can be separated into its components, we know that both sides must be equal to the constant. As a result, we have obtained.

$$Y' + \lambda Y = 0,$$

$$\left(\frac{X''}{X}\right)' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right)' = 0.$$

After integrating the equation with respect to x, we arrive at the following.

$$\frac{X''}{X} + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right) = -\beta,$$

where is a constant that has to be worked out. When we plug equation into the first equation, we get the result that we're looking for.

$$X'' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}X' + \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}X = 0.$$

When we compare equations, it is quite evident that.

$$\beta = \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}.$$

We have just gone through the requirements that must be met in order for a certain partial differential equation to be separable. Now that we have reached this point, let's have a look at the boundary conditions that are in play. There are several variants of boundary conditions to choose from. The ones that appear most frequently in problems of applied mathematics and mathematical physics include.

In addition to these three boundary requirements, which are also referred to as the first, second, and third boundary conditions, there are other conditions, such as the Robin condition, in which one boundary condition is specified on a piece of a boundary and another boundary condition is supplied on the rest of the border. When we finally get down to solving issues, we are going to take into account a wide range of boundary conditions. It is important to use a coordinate system that is appropriate for a border in order to distinguish boundary conditions like the ones that were stated above. For example, we choose the Cartesian coordinate system for a rectangular area such that the border may be defined by the lines of the coordinate system.

The study of partial differential equations provides the foundation for much of contemporary science, engineering, and mathematics. A partial differential equation is an equation with partial derivatives that implicitly deOnes a function of two or more variables.

thereafter, under appropriate circumstances $u(x,t)$ is a solution to the equation describing heat.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where k represents a constant value. Consider, for a further illustration, that if is the displacement of a string at time t, then the string's vibration is likely to meet the one-dimensional wave equation for a constant, which is.

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

When an application requires the use of a partial differential equation, our objective is often to find a solution to the equation. According to this definition, a function is considered a solution to a partial differential equation if it can be implicitly deduced from the equation. That is, a function that satisfies the equation is considered to be a solution.

The method known as separation of variables may be used to acquire solutions to a large number of partial differential equations, however this method will not always be successful. It is predicated on the observation that if f (x) and g (t) are functions of the independent variables x and t, respectively, and if it is true that if, then it follows that if.

$$f(x) = g(t)$$

Consequently, there must exist a constant in which $f(x) = \lambda$ and $g(t) = \lambda.$ (The evidence is really easy to understand, given that).

$$\begin{aligned} \frac{\partial}{\partial x}f(x) &= \frac{\partial}{\partial x}g(t) = 0 \implies f'(x) = 0 \implies f(x)\ \ constant \\ \frac{\partial}{\partial t}g(t) &= \frac{\partial}{\partial t}f(x) = 0 \implies g'(t) = 0 \implies g(x)\ \ constant) \end{aligned}$$

In the process of separating out variables, the starting point is always the assumption that the answer is in the separated form.

$$u(x,t) = X(x)\,T(t)$$

After that, we plug the separated form back into the equation and, if it's at all practicable, shift the x-terms to one side of the equation and the t-terms to the other. If it is not feasible, then this approach will not work, and hence, the partial differential equation will not be separable. If it is not possible, then this method will not work. Once the two sides of the equation have been separated, they must remain constant in order to satisfy the requirements of an ordinary differential equation, in which X(x) is a function of some value of x and T(t) is a function of some value of t. Because any temperature u(x, t) of this type will keep its fundamental "shape" for various values of time t, the solutions are straightforward and easy to understand. The separation of variables made it possible to simplify the difficulty of solving the partial differential equation to the task of solving two ordinary differential equations (ODEs) instead: one second order ODE involving the independent variable x, and one first order ODE involving t. After that, these ODEs are solved by applying the supplied beginning conditions and boundary conditions.

Let's apply this strategy to a concrete issue so that we can better understand how it works. Take into consideration the IBVP that follows.

$$\begin{aligned} \text{PDE:} \quad & u_t = \alpha^2 u_{xx}, \quad 0 \le x \le L,\ 0 < t < \infty, \\ \text{BC:} \quad & u(0,t) = 0 \ \ u(L,t) = 0, \quad 0 < t < \infty, \\ \text{IC:} \quad & u(x,0) = f(x), \quad 0 \le x \le L. \end{aligned}$$

Let's assume that equation has solutions, and those solutions have the form.

$$\boxed{u(x,t) = X(x)T(t),}$$

where X is determined only by the value of x, and T is determined solely by the value of t. Take not that.

$$u_t = X(x)T'(t) \quad \text{and} \quad u_{xx} = X''(x)T(t).$$

Now, let's try plugging in this phrase into: $u_t = \alpha^2 u_{xx}$ by identifying the different factors, we are able to.

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}.$$

HEAT EQUATION

Given that it is only possible for a function of t to be equivalent to a function of x when both functions are constant. Thus.

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = c$$

in the presence of some constant c. This results in the two ODEs that are shown below:

$$T'(t) - \alpha^2 c T(t) = 0,$$
$$X''(x) - cX(x) = 0.$$

As a result, the challenge of resolving the PDE may now be broken down into the solution of the two ODEs.

OBJEACTIVES

1. The Study Basic Concepts Examples and One Method of Separation of Variables.
2. The Study Involve Second-Order Partial Differential Equations in Two Independent Variables.

Euler's Differential Equation

Euler's equation is the most straightforward example of a linear variable coefficient second order ODE:

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + c\,y = 0$$

With the help of the ansatz, we attempt to find a solution:

$$y = x^r$$

It is possible for us to utilize the quadratic formula in order to acquire (in general) a pair of complex conjugate roots denoted by r1 and r2. Therefore, the answer in its general form may be written.

$$y = c_1\, x^{r_1} + c_2\, x^{r_2}$$

On the other hand, in the general situation when r1 and r2 both have a complex form, say $r_1 = \mu + i\nu,$ We get a hold of the form.

$$y = c_1\, x^{\mu+i\nu} + c_2\, x^{\mu-i\nu} = x^{\mu}(c_1 x^{i\nu} + c_2 x^{-i\nu})$$

Power Series Solutions

Therefore, we are aware of the solution to Euler's equation. What about linear differential equations that don't take the form of Euler's equation and have variable coefficients? A power series would be a logical place to start looking for a solution, therefore that would be a natural approach:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

where the ci coefficients may be discovered looking for them. An example is the most effective way to demonstrate the process.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

and collect terms:

$$2c_2 + c_0 + x(6c_3 + 2c_1) + x^2(12c_4 + 3c_2) + x^3(20c_5 + 4c_3)\cdots$$

The next thing that we need is the coefficient for every power of. x^n to vanish, giving:

$$\begin{aligned} 2c_2 + c_0 &= 0 \\ 6c_3 + 2c_1 &= 0 \\ 12c_4 + 3c_2 &= 0 \\ 20c_5 + 4c_3 &= 0 \end{aligned}$$

The Method of Frobenius

The technique of power series is extended by the Frobenius approach, which searches for a solution in the form of a "generalized power series":

$$y = x^r(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots + c_nx^n + \cdots)$$

It all boils down to the power series.

$$2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (-1+x)y = 0$$

We provide alternatives and compile terminology:

$$(2r^2 + r - 1)c_0x^r + [(2r^2 + 5r + 2)c_1 + c_0]x^{r+1} + [(2r^2 + 9r + 9)c_2 + c_1]x^{r+2} + \cdots = 0$$

$$\begin{aligned} (2r^2 + r - 1)c_0 &= 0 \\ (2r^2 + 5r + 2)c_1 + c_0 &= 0 \\ (2r^2 + 9r + 9)c_2 + c_1 &= 0 \end{aligned}$$

$$[2r^2 + (4n+1)r + (n^2 + n - 1)]c_{n+1} + c_n = 0$$

consist of the recurrence connection:

$$c_{n+1} = -\frac{c_n}{(2r + 2n - 1)(r + n + 1)}, \qquad n = 1, 2, 3, \cdots$$

We provide individual consideration to each of these R-values, using as our basis for doing so:

$$y = f(x) = \frac{1}{x} + 1 - \frac{x}{2} + \frac{x^2}{18} - \frac{x^3}{360} + \frac{x^4}{12600} - \frac{x^5}{680400} + \cdots$$

For $r=\frac{1}{2}$ we get:

$$y = g(x) = \sqrt{x}\left(1 - \frac{x}{5} + \frac{x^2}{70} - \frac{x^3}{1890} + \frac{x^4}{83160} - \frac{x^5}{5405400} + \cdots\right)$$

We provide alternatives and compile terminology:

$$c_0 x^r + [(r^2 - r)c_0 + c_1]x^{r+1} + [(r^2 + r)c_1 + c_2]x^{r+2} + [(r^2 + 3r + 2)c_2 + c_3]x^{r+3} + \cdots + = 0$$

$$\begin{aligned} c_0 &= 0 \\ (r^2 - r)c_0 + c_1 &= 0 \\ (r^2 + r)c_1 + c_2 &= 0 \\ (r^2 + 3r + 2)c_2 + c_3 &= 0 \end{aligned}$$

$$(r+n)(r+n-1)c_n + c_{n+1} = 0$$

consist of the recurrence connection:

$$c_{n+1} = -(r+n)(r+n-1)c_n, \qquad n = 1, 2, 3, \cdots$$

To summarize, we have shown that the approach of power series is effective for solving certain equations, but for other equations, that method is ineffective, but the more general method of Frobenius is effective. An illustration of how the Frobenius approach can't work has just been shown to us.

We want a classification technique that will inform us which equations we are able to reliably solve by using either of these two approaches.

Ordinary Points and Singular Points

First, let's take a look at the broad category of linear second-order ODEs represented by the form:

$$A(x)\, y'' + B(x)\, y' + C(x)\, y = 0$$

in which $A(x)$, $B(x)$ and $C(x)$ have power series expansions about $x=0$:

$$A(x) = A_0 + A_1 x + A_2 x^2 + \cdots$$
$$B(x) = B_0 + B_1 x + B_2 x^2 + \cdots$$
$$C(x) = C_0 + C_1 x + C_2 x^2 + \cdots$$

Orthogonality

It is necessary for us to make use of orthogonality of the linked eigenfunctions so that we may finish our treatment of issue "B." The orthogonal eigenfunctions X(x) were able to solve the following boundary value issue when applied to the "A" problem scenario.

$$X'' + \lambda X = 0 \quad \text{with the B.C.} \quad X(0) = 0, \quad X(L) = 0$$

We addressed this difficulty in order to achieve the results we needed to demonstrate orthogonality.

$$X_n(x) = \sin \lambda_n x = \sin \frac{n\pi x}{L}$$

Afterwards, in order to demonstrate our point, we employed the characteristics of trig functions.

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0 \qquad n \neq m$$

The orthogonality of Bessel functions cannot be shown using a similar approach, which is unfortunately the case. Instead, we are going to use an alternative strategy known as the Sturm-Liouville theory. We provide a demonstration of the procedure.

$$X_n'' + \lambda_n X_n = 0 \quad \text{with the B.C.} \quad X_n(0) = 0, \quad X_n(L) = 0$$

We want to prove that.

$$\int_0^L X_m X_n dx = 0 \quad \text{for} \quad n \neq m$$

By using integration by parts and the B.C. in method, the goal at this point is to make the first term disappear. Take into consideration the first few weeks of the first semester:

$$\int_0^L X_m X_n''\, dx = [X_m X_n']_0^L - \int_0^L X_m' X_n'\, dx$$

The integrated terms vanish due to the B.C. $X_m(0)=X_m(L)=0.$ Thus.

$$\int_0^L X_m X_n'' \, dx = -\int_0^L X_m' X_n' \, dx$$

Similarly,

$$\int_0^L X_n X_m'' \, dx = -\int_0^L X_n' X_m' \, dx$$

Therefore eq. becomes.

$$(\lambda_n - \lambda_m) \int_0^L X_n X_m \, dx = 0 \quad \Rightarrow \quad \int_0^L X_n X_m \, dx = 0 \quad \text{if} \quad n \neq m$$

Without having to create anything, we were able to demonstrate that the eigenfunctions are orthogonal. These two methods couldn't be more unlike to one another.

CONCLUSION

Equations with three or more independent variables may be solved using a method that is similar to this one. Nevertheless, the issue of whether or not a partial differential equation can be broken down into two or more ordinary differential equations is not at all an easy one to answer. Despite the existence of this concern, the approach is commonly used in the process of finding solutions to a significant number of initial boundary-value problems. Another name for this approach to problem solving is the Fourier technique, and it's also sometimes called the method of eigenfunction expansion. Therefore, the process that was stated above leads to the significant concepts of eigenvalues, eigenfunctions, and orthogonality, all of which are quite generic and strong when it comes to dealing with linear issues. The application of this strategy to real-world problems is shown by the following instances in a broader sense.

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